Decoding solar wind turbulence with nonrelativistic linearized Vlasov-Maxwell instability theory

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5 August 2010

Topics:

1. Introduction
2. Dispersion relations of parallel fluctuations
3. Observed solar wind fluctuations
4. Weakly amplified solutions
5. Weakly propagating solutions
6. Summary and conclusions

References:

R. Schlickeiser, Linear theory of temperature anisotropy instabilities in magnetized pair plasmas, The Open Plasma Physics Journal 3 (2010) 1

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1. Introduction

The solar wind plasma is the only cosmic plasma where detailed in-situ satellite observations of plasma properties are available (Bale et al. 2009). As other dilute cosmic plasmas have similar densities, temperatures and magnetic fields as the solar wind, the physical processes of turbulence generation probably apply to all other dilute cosmic plasmas.

Ten years (long-time average) of WIND/SWE data (see Fig. 1) have demonstrated that the proton and electron temperature anisotropies $A = T_\perp / T_\parallel$ are bounded by mirror and firehose instabilities at large values $\beta_\parallel \geq 1$ of the parallel plasma beta $\beta_\parallel = 8\pi n k_B T_\parallel / B_0^2 = P_{\mathrm{thermal},\parallel} / P_B$.

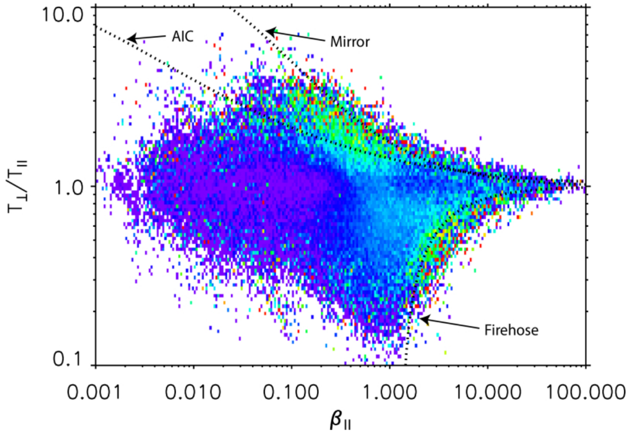

Figure 1: Magnitude of observed magnetic fluctuations $|\delta B|$ (from Bale et al. 2009)

In the parameter plane defined by the temperature anisotropy $A = T_\perp / T_\parallel$ and the parallel plasma beta $\beta_\parallel$, stable plasma configuration are only possible within a rhomb-like configuration around $\beta_\parallel \simeq 1$, whose limits at large $\beta_\parallel \gg 1$ are defined by the threshold conditions for the mirror and firehose instabilities. If a plasma would start with parameter values outside this rhomb-like configuration, it immediately would generate fluctuations via the mirror and firehose instabilities, which quickly relax the plasma distribution into the stable regime within the rhomb-configuration.

What happens at small parallel plasma betas?

In order to understand the confinement limits also at small values of the parallel plasma beta $\beta_{\|} < 1$ we analyzed rigorously the full **linear dispersion relation** in a collisionless homogenous plasma with **anisotropic** $(A \neq 1)$ **bi-Maxwellian particle velocity distributions** for electromagnetic fluctuations with real wave vectors ($\vec{k} \times \vec{B}_0 = 0$) parallel to the uniform background magnetic field $\vec{B}_0$ and complex frequency

$$\omega(k) = \omega_R(k) + \imath\gamma(k), \quad \delta\vec{B}(z,t) \propto e^{\imath(kz-\omega_R t)} e^{\gamma t} = e^{\imath k(z-Rt)} e^{Skt} \qquad (1)$$

Two types of fluctuations:

(1) **Weakly amplified** solutions with $\gamma \ll \omega_R$.

(2) **Weakly propagating** solutions with $\omega_R \ll \gamma$ including aperiodic solutions with $\omega_R = 0$.

2. Dispersion relations of parallel fluctuations

2.1. Basic equations

The theoretical problem is well posed: for a nonzero background magnetic field strength the nonrelativistic dispersion relations for right-handed (RH) and left-handed (LH) polarized transverse fluctuations with wave vectors $\vec{k} \times \vec{B} = 0$ in a thermal electron-proton plasma are (Gary 1993)

$$0 = D_{RH,LH}(k,\omega) = \omega^2 - k^2c^2 +$$
$$\sum_{a=p,e} \omega_{p,a}^2 \left[\frac{\omega}{\sqrt{2}ku_{a,\parallel}} Z\left(\frac{\omega \pm \Omega_a}{\sqrt{2}ku_{a,\parallel}}\right) + \frac{1}{2}\left(1 - A_a\right) Z^{'}\left(\frac{\omega \pm \Omega_a}{\sqrt{2}ku_{a,\parallel}}\right)\right] = 0, \quad (2)$$

where we sum over a proton (p)-electron (e) plasma. and where $\omega_{p,e}$ denotes the electron plasma frequency, $u_{a,\parallel} = (k_B T_{a,\parallel}/m_a)^{1/2}$ is the parallel thermal velocity of component a, $\Omega_a = e_a B/(m_a c)$ is the non-relativistic gyrofrequency, and $A_a = T_{a,\perp}/T_{a,\parallel}$ is the temperature anisotropy of component a, where the directional subscripts refer to directions relative to the background magnetic field. The dispersion relations (2) allow for different values of the proton and electron parallel temperatures and temperature anisotropies.
$Z(x)$ and $Z^{'}(x)$ denote the plasma dispersion function (Fried and Conte 1961) and its derivative

$$Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \, \frac{e^{-t^2}}{t - x} \quad (3)$$

We use the asymptotic expansions

$$Z(x) \simeq \imath \pi^{1/2} e^{-x^2} - 2x \left[1 - \frac{2x^2}{3} \right], \ |x| \ll 1 \tag{4}$$

and

$$Z(x) \simeq \imath \sigma \pi^{1/2} e^{-x^2} - \frac{1}{x} \left[1 + \frac{1}{2x^2} + \frac{3}{4x^4} \right], \ |x| \gg 1 \tag{5}$$

where $\sigma = 0$ if $\Im(x) > 0$, $\sigma = 1$ if $\Im(x) = 0$ and $\sigma = 2$ if $\Im(x) < 0$.
It is convenient to introduce the complex phase speeds

$$f = \frac{\omega}{kc} = \frac{\omega_R + \imath\gamma}{kc} = R + \imath S, \quad R = \frac{\omega_R}{kc}, \quad S = \frac{\gamma}{kc}, \tag{6}$$

the plasma frequency phase speed

$$w = \frac{\omega_{p,e}}{kc}, \tag{7}$$

and the absolute value of the electron gyrofrequeny phase speed

$$b = \frac{|\Omega_e|}{kc}, \tag{8}$$

where $|\Omega_e| = eB/m_e c$. $\mu = m_p/m_e = 1836$ is the mass ratio, and

$$\Theta_e \equiv \left(\frac{2k_B T_{e,\parallel}}{m_e c^2} \right)^{1/2}, \ \ \Theta_p \equiv \left(\frac{2k_B T_{p,\parallel}}{m_p c^2} \right)^{1/2} \tag{9}$$

We discuss high density plasmas with $\omega_{p,e} \gg |\Omega_e|$, corresponding to $w \gg b$, which applies to nearly all astrophysical plasmas. These plasmas are dense enough that the electron plasma frequency is much larger than the electron gyrofrequency, but small enough that elastic Coulomb collisions can be neglected. The two dispersion relations (2) then read ($f = R + \imath S$)

$$0 = \frac{D_{RH,LH}(k,f)}{k^2c^2} = \Lambda_{RH,LH}(k,f) = f^2 - 1 + \frac{w^2}{\mu}\left[\frac{f}{\Theta_p} Z\left(\frac{f \pm \frac{b}{\mu}}{\Theta_p}\right) + \frac{1}{2}\left(1 - A_p\right) Z'\left(\frac{f \pm \frac{b}{\mu}}{\Theta_p}\right)\right] + w^2\left[\frac{f}{\Theta_e} Z\left(\frac{f \mp b}{\Theta_e}\right) + \frac{1}{2}\left(1 - A_e\right) Z'\left(\frac{f \mp b}{\Theta_e}\right)\right] \quad (10)$$

The symmetry $\Lambda(-k,f) = \Lambda(k,f)$ of both dispersion relations allows us to consider only positive values of the wavenumber $k > 0$. To simplify the analysis we consider only equal parallel temperature plasmas ($T_{e,\parallel} = T_{p,\parallel}$) so that $\Theta_e = \Theta$ and $\Theta_p = \Theta/\mu^{1/2}$.

The dispersion relations (10) can be separated into real and imaginary parts

$$\Lambda(R,S) = \Re\Lambda(R,S) + \imath\Im\Lambda(R,S) = 0 \quad (11)$$

2.2. Weak amplification limit

In the weak amplification approximation, we equate the real and imaginary parts to zero, make a Taylor-expansion around $S = 0$, and use the Cauchy-Riemann relations to obtain

$$\Re\Lambda(R, S = 0) = 0, \quad S(k) = -\frac{\Im\Lambda(R, S = 0)}{\frac{\partial\Re\Lambda(R,S=0)}{\partial R}} \tag{12}$$

2.3. Weak propagation limit

In the weakly propagating limit ($R \ll |S|$) we Taylor-expand around $R = 0$ to obtain in a similar way

$$\Re\Lambda(R = 0, S) = 0, \quad R(k) = \frac{\Im\Lambda(R = 0, S)}{\frac{\partial\Re\Lambda(R=0,S)}{\partial S}} \tag{13}$$

3. Observed solar wind fluctuations

The solar wind magnetic fluctuations measured by Bale et al. (2009) near 1 AU have wavenumbers

$$k \simeq \alpha/\rho_p \tag{14}$$

with $\alpha = 0.56 \pm 0.32$ and the thermal proton gyroradius $\rho_p = 4.23 \cdot 10^6 T_5^{1/2} B_4^{-1}$ cm, where we adopt an interplanetary magnetic field value $B = 10^{-4} B_4$ Gauss and a temperature $T = 10^5 T_5$ K. With the solar wind particle density $n_e = 10^2 n_2$ cm^{-3} we find that the plasma frequency phase speed (7)

$$w = \frac{79.5}{\alpha} \frac{(T_5 n_2)^{1/2}}{B_4} = \frac{142}{1 \pm 0.57} \frac{(T_5 n_2)^{1/2}}{B_4} \tag{15}$$

covers the interval

$$90 \frac{(T_5 n_2)^{1/2}}{B_4} \leq w \leq 330 \frac{(T_5 n_2)^{1/2}}{B_4} \tag{16}$$

The electron gyrofrequency phase speed (8)

$$b = 3.13 \cdot 10^{-3} w \frac{B_4}{n_2^{1/2}} \tag{17}$$

is indeed much smaller than the electron plasma frequency phase speed w justifying the high density plasma approximation $w \gg b$.

4. Weakly amplified solutions

For weakly damped or amplified fluctuations

$$0 = \Re\Lambda_{RH,LH}(R, S = 0) = R^2 - 1$$
$$+w^2\Big[\frac{R}{\Theta}\Re Z\left(\frac{R \mp b}{\Theta}\right) + \frac{1-A_e}{2}\Re Z'\left(\frac{R \mp b}{\Theta}\right)$$
$$+\frac{R}{\Theta\mu^{1/2}}\Re Z\left(\frac{\mu^{1/2}}{\Theta}\left[R \pm \frac{b}{\mu}\right]\right) + \frac{1-A_p}{2\mu}\Re Z'\left(\frac{\mu^{1/2}}{\Theta}\left[R \pm \frac{b}{\mu}\right]\right)\Big] \qquad (18)$$

and

$$\Im\Lambda_{RH,LH}(R, S = 0) = \frac{w^2}{2}\Big[\frac{1-A_p}{\mu}\Im Z'\left(\frac{\mu^{1/2}}{\Theta}\left[R \pm \frac{b}{\mu}\right]\right)$$
$$+(1-A_e)\Im Z'\left(\frac{R \mp b}{\Theta}\right)\Big] + \frac{w^2 R}{\Theta}\left[\frac{1}{\mu^{1/2}}\Im Z\left(\frac{\mu^{1/2}}{\Theta}\left[R \pm \frac{b}{\mu}\right]\right) + \Im Z\left(\frac{R \mp b}{\Theta}\right)\right], \qquad (19)$$

which have to be investigated for positive values of $R \geq 0$.
With the asymptotic expansions (4) and (5), depending on the absolute values of the arguments of the plasma dispersion function being small or large compared to unity, polynomial forms of the dispersion relations (18)-(19) can be derived.

4.1. LH-polarized Alfven-proton-cyclotron and RH-polarized Alfven-Whistler-electron cyclotron branches

For $\beta_\parallel \ll (w^2/\mu)$ the asymptotic expansion (5) yield to lowest order in $\Theta^2 \ll 1$ with the scaling $R = bx$

$$0 = \Re\Lambda_{RH,LH}(x, S = 0) = \left[b^2 + \frac{(1+\mu)w^2}{(1 \pm \mu x)(1 \mp x)}\right] x^2 - 1$$
$$\pm \frac{\beta_\parallel}{2}\left(\frac{A_e x \pm (1 - A_e)}{(1 \mp x)^3} - \frac{A_p \mu x \mp (1 - A_p)}{(1 \pm \mu x)^3}\right) \quad (20)$$

In different limits the solutions of Eqs. (20) describe Alfven waves, Whistler waves, cyclotron waves and electromagnetic waves.

4.2. Small plasma beta $\beta_\parallel \ll 1$

For subluminal solutions $R \ll 1$ and small parallel plasma beta $\beta_\parallel \ll 1$:

$$0 = \Re\Lambda_{RH,LH}(x, S = 0) \simeq \frac{(1+\mu)w^2 x^2}{1 \pm (\mu - 1)x - \mu x^2} - 1 \quad (21)$$

yielding the solutions

$$x_{RH,LH} = \frac{1}{2(1+w^2)}\left[\sqrt{1 + \frac{4(1+w^2)}{\mu}} \pm 1\right], \quad (22)$$

For $w \gg \sqrt{(\mu/4)-1} = 21.4$, corresponding to small wavenumbers $k \ll \omega_{p,e}/21.4c$, we obtain the Alfven wave solution

$$x_{RH,LH} \simeq \frac{1}{\sqrt{\mu(1+w^2)}} \simeq \frac{1}{\mu^{1/2}w}, \tag{23}$$

Consequently, at the measured $w \in [90, 330]$ of Bale et al. (2009) Alfven waves are the only weakly amplified modes contributing. Allowing also large plasma beta values $\beta_\parallel \leq 11(w/142)^2$ the Alfvenic dispersion relations generalizes to

$$x_{RH,LH} \simeq \frac{1}{\sqrt{1+\mu w}}\sqrt{1+(A-1)\beta_\parallel}, \tag{24}$$

requiring $A = (A_e + A_p)/2 > 1 - \beta_\parallel^{-1}$. For small plasma betas ($\beta_\parallel \leq 1$) this condition is always fulfilled.

4.3. Growth rate and instability condition

The growth/damping rate then is

$$S_{RH,LH} = -\frac{\pi^{1/2}(1+\mu)b^2w^4x^3}{\Theta[2\pm(\mu-1)x]}$$

$$\times\left(\frac{1}{\mu^{1/2}}\left[A_p\left[x\pm\frac{1}{\mu}\right]\mp\frac{1}{\mu}\right]e^{-\frac{\mu b^2}{\Theta^2}(x\pm\frac{1}{\mu})^2}+[A_e(x\mp 1)\pm 1]\,e^{-\frac{b^2(x\mp 1)^2}{\Theta^2}}\right), \quad (25)$$

For isotropic ($A_p = A_e = 1$) temperatures all modes of the RH and LH branches are damped in agreement with Brinca's (1990) general theorem.

For marginal instability $S_{RH,LH} > 0$, yielding for $A_p = A_e = A_0$ with $W_\pm(x) = \frac{w^2}{2\beta_\parallel}\left[\pm 4x + (\mu-1)(x^2-\frac{1}{\mu})\right]$ the instability condition

$$\pm\left(1-\frac{1}{A_0}\right)\left[(\mu^{3/2}-1)+(\mu^{3/2}+1)\tanh(W_\pm\right]$$

$$> \mu x\left[\mu^{1/2}+1+(\mu^{1/2}-1)\tanh(W_\pm)\right]. \quad (26)$$

4.4. Left-handed and right-handed polarized Alfven waves

For Alfven waves the instability condition (26) becomes

$$\pm\left(1-\frac{1}{A_0}\right)\left[(\mu^{3/2}+1)\tanh\left[\frac{w^2}{2\beta_{\|}}\right]-(\mu^{3/2}-1)\right]$$
$$> x\mu\left[(\mu^{1/2}+1)-(\mu^{1/2}-1)\tanh\left[\frac{w^2}{2\beta_{\|}}\right]\right], \tag{27}$$

Because $w^2/2\beta_{\|} >> \mu/2 = 918$ we set $\tanh\left[\frac{w^2}{2\beta_{\|}}\right] = 1$, providing with the dispersion relation (24) for LH polarized Alfven waves

$$\left(1-\frac{1}{A_0}\right) > \mu x = \frac{1}{h}\sqrt{1+(A_0-1)\beta_{\|}},\ \ h=\frac{w}{\mu^{1/2}} \tag{28}$$

which can only be fulfilled for $A_0 > 1$. h is the proton plasma frequency phase speed.
For RH polarized Alfven waves

$$\left(\frac{1}{A_0}-1\right) > \mu x = \frac{1}{h}\sqrt{1-(1-A_0)\beta_{\|}}, \tag{29}$$

which can only be fulfilled for $A_0 < 1$. Recall the constraint $A_0 > 1-\beta_{\|}^{-1}$ here for $\beta_{\|} > 1$. The two instability conditions are shown in Fig. 2.

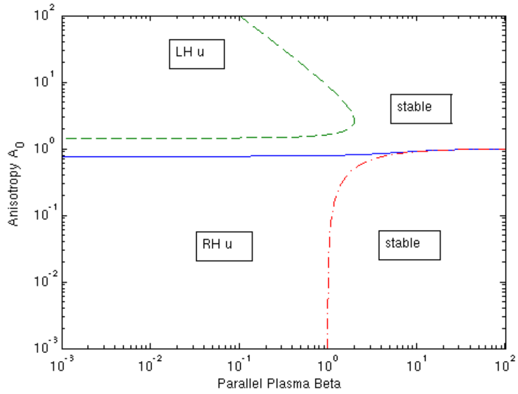

Figure 2: Anisotropy diagram of LH (dashed curve) and RH (full curve) polarized Alfven waves calculated for $w = 142$ corresponding to a proton plasma frequency spase speed $h = 3.31$. Unstable regions are marked by "u". The dot-dashed curve illustrates the constaint $A_0 > 1 - \beta_{\parallel}^{-1}$.

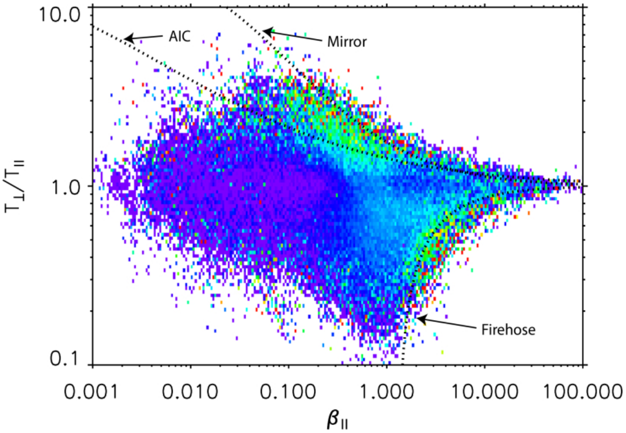

Figure 3: Magnitude of observed magnetic fluctuations $|\delta B|$ (from Bale et al. 2009)

- Unstable LH and RH polarized Alfven waves explain the observed confinement limits of solar wind turbulence at small parallel plasma beta, especially at large temperature anisotropies $A > 1$.

- Anisotropy diagram controlled by single parameter: the proton plasma frequency phase speed $h = w/\mu^{1/2} \simeq 3.31$ for Bale et al. (2009) observations.

- For RH polarized waves $A_0 < h/(h+1) = 0.77$ for $\beta_{\|} \to 0$.

- For LH polarized waves $A_0 > h/(h-1) = 1.43$ for $\beta_{\|} \to 0$.

- Maximum parallel plasma beta $\beta_{\|,max} = 1.968$ of unstable LH polarized Alfven waves at $A_0 = y_1/(1-y_1) = 2.634$ with

$$y_1 = h^{2/3}\left(\left[\sqrt{1+\frac{h^2}{27}}+1\right]^{1/3} - \left[\sqrt{1+\frac{h^2}{27}}-1\right]^{1/3}\right) \tag{30}$$

- Situation at small temperature anisotropies $A < 1$ not in accord with Bale-observations. (Repeat the analysis for different proton and electron anisotropies and/or parallel temperatures. Do the corresponding analysis for perpendicular wave vectors.)

5. Weakly propagating solutions

For weakly propagating fluctuations with positive $S > 0$ we obtain

$$0 = \Re\Lambda_{RH,LH}(R=0,S) = -S^2 - 1 - \frac{w^2 S}{\Theta}\Big[\frac{1}{\mu^{1/2}}\Im Z\left(\frac{\mu^{1/2}}{\Theta}\left[\imath S \pm \frac{b}{\mu}\right]\right) + \Im Z\left(\frac{\imath S \mp b}{\Theta}\right)\Big]$$

$$+ \frac{w^2}{2}\left[\frac{1-A_p}{\mu}\Re Z'\left(\frac{\mu^{1/2}}{\Theta}\left[\imath S \pm \frac{b}{\mu}\right]\right) + (1-A_e)\Re Z'\left(\frac{\imath S \mp b}{\Theta}\right)\right] \qquad (31)$$

and

$$\Im\Lambda_{RH,LH}(R=0,S) = \frac{w^2}{2}\Big[\frac{1-A_p}{\mu}\Im Z'\left(\frac{\mu^{1/2}}{\Theta}\left[\imath S \pm \frac{b}{\mu}\right]\right) + (1-A_e)\Im Z'\left(\frac{\imath S \mp b}{\Theta}\right)\Big]$$

$$+ \frac{w^2 S}{\Theta}\left[\frac{1}{\mu^{1/2}}\Re Z\left(\frac{\mu^{1/2}}{\Theta}\left[\imath S \pm \frac{b}{\mu}\right]\right) + \Re Z\left(\frac{\imath S \mp b}{\Theta}\right)\right] \qquad (32)$$

Again polynomial dispersion relations can be derived for small and large absolute values of the arguments of Z.

Four weakly propagating solutions are found which are identified as

- mirror fluctuations,
- IMW (intermediate magnetized Weibel) fluctuations ,
- electron cyctronic fluctuations (newly found),
- firehose fluctuations.

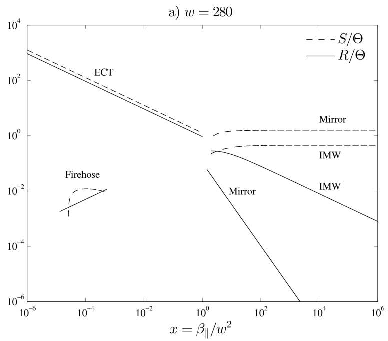

Figure 4: Real (R) and imaginary (S) phase speeds of weakly propagating fluctuations as a function of parallel plasma beta calculated for $w = 280$.

5.1. Comments on weakly propagating solutions

- In contrast to pair plasmas, all modes are weakly propagating in a finite magnetic field, as they have a finite real phase speed $R \neq 0$.
- In the limit of an unmagnetized plasma the mirror and IMW fluctuations approach the dispersion relations of the aperiodic hot and cool Weibel fluctuations, whereas parallel firehose and electron cyctronic fluctuations do not exist.
- The mirror and IMW fluctuations occur at large plasma beta values $\beta_\| > w^2$, whereas the electron cyctronic fluctuations occur at small values of $\beta_\| < w^2$.
- The firehose fluctuations are restricted to the range $1 \leq \beta_\| \leq w^2/\mu = h^2$ and require $w > \mu^{1/2}$ $(h > 1)$.

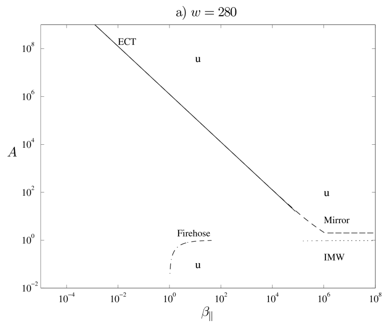

Figure 5: Anisotropy diagram of weakly propagating fluctuations calculated for $w = 280$.

At large parallel plasma beta and small anisotropies $A < 1$ unstable firehose fluctuations operate, whereas at large parallel plasma beta and large anisotropies $A > 1$ mirror, IMW and electron cyctronic explain the observed confinement limits of solar wind turbulence.

6. Summary and conclusions

We rigorously studied the dispersion relations of weakly amplified and weakly propagating transverse fluctuations with wave vectors $\vec{k}\times\vec{B}_0 = 0$ in an anisotropic bi-Maxwellian magnetized proton-electron plasma.

- The Alfven instability is the only weakly amplified solution, whereas the four weakly propagating solutions are the mirror, electron cyclotronic, firehose and intermediate magnetized Weibel fluctuations, respectively.
- The four weakly propagating solutions are weakly propagating in a finite magnetic field with finite real phase speeds $R \propto b > 0$.
- In an unmagnetized plasma the mirror and IMW fluctuations approach the dispersion relations of the aperiodic hot and cool Weibel fluctuations, whereas parallel firehose and electron cyctronic fluctuations do not exist.
- In agreement with Brinca's (1990) general theorem on the electromagnetic stability of isotropic plasma populations none of these modes can be excited for isotropic plasma distributions ($A = 1$).
- For equal parallel electron and proton temperatures, a general analytical instability condition for weakly amplified fluctuations is derived for equal values of the electron and proton temperature anisotropies.
- The conditions, for which the weakly amplified LH-handed and RH polarized Alfven waves can be excited, are derived. Unstable LH and RH

polarized Alfven waves explain the observed confinement limits of solar wind turbulence of Bale et al. (2009) at small parallel plasma beta.

- At large parallel plasma beta and small anisotropies $A < 1$ unstable firehose fluctuations operate, whereas at large parallel plasma beta and large anisotropies $A > 1$ mirror, IMW and electron cyctronic explain the observed confinement limits of solar wind turbulence.

- Apparently the combined action of all five instabilities account for the observations.